Tate's thesis and the Riemann-Roch theorem

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Tate's thesis

In his PhD Thesis (1950) John Tate proved meromorphic continuation and functional equation of the Dedekind zeta function,i.e. a zeta function associated to a number field, using harmonic analysis tools, more precisely Poisson summation formula. It is a very elegant reformulation of Hecke's work.

Poisson summation formula

The Poisson summation formula also implies the Riemann-Roch theorem. Moreover it does it uniformly for the geometric and the arithmetic cases as well as the global and the relative cases (through renormalisation of the Haar measure on $\mathbb{A}_k$).

By analogy with the function field of the projective line $\mathbb{F}_p(t)$ one can see the field of rational numbers $\mathbb{Q}$ as the field of rational functions on the arithmetic projective line.

We call finite extensions of $\mathbb{Q}$ and $\mathbb{F}_p(t)$ global fields.

Remark

Even though the case of $\mathbb{C}(t)$ is the most classical we don't work with it as $\mathbb{C}(t)$ is not locally compact as an abelian group which will be crucial for our applications.

Norms

A norm on a field k is function

$$| \ | : k \to \mathbb{R}_{\geq 0}$$

such that for $a, b \in k$

- $|a| = 0$ if and only if $a = 0$,
- $|ab| = |a||b|$,
- $|a + b| \leq |a| + |b|$ (triangle inequality).

Non-archimedean

A norm is called non-archimedean if

$$|a+b| \leq \max(|a|,|b|).$$

It is easy to see that this condition is equivalent to not satisfying the axiom of Archimedes. i.e:

$$|a| < |b| \text{ then there exist } n \in \mathbb{Z}_{>0} \text{ such that } n|a| > |b|.$$

Otherwise it is called archimedean.

p-adic norms

Definition

For $a \in \mathbb{Q}$ one defines

$$|a|_p = p^{-v(a)}$$

where $v(a)$ is the power of p that divides a.

If

$a = \frac{b}{c}$ with $b, c \in \mathbb{Z}$

then

$$v(a) = v(b) - v(c).$$

$v : \mathbb{Q} \to \mathbb{Z}$ - valuation on $\mathbb{Q}$.

infinite prime

Definition

We denote the standard norm induced from $\mathbb{R}$ as

$$|a|_\infty$$

and call ∞ an infinite prime of $\mathbb{Z}$.

Equivalence of norms

Two norms $| \ |_1$, $| \ |_2$ on k are said to be equivalent if there exists a real number $c > 0$ such that

$$|a|_1 = |a|_2^c \text{ fora all } a \in k$$

Ostrowski's theorem

Theorem (Ostrowski)

The only absolute values (up to equivalence) on $\mathbb{Q}$ are

$|\ |_p$ *where p is a finite prime,*

$|\ |_\infty$ *where ∞ is an infinite prime.*

Analogy with $\mathbb{P}^1_{\mathbb{C}}$

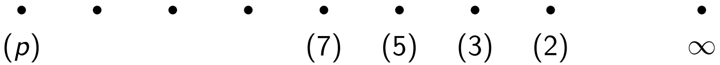
Note that for for the absolute values on the function field of $\mathbb{P}^1(\mathbb{F}_p)$ are in $1:1$ correspondence with points of $\mathbb{P}^1(\mathbb{F}_p)$ (up to equivalence of absolute values). Indeed, the absolute value

functions on any $\mathbb{F}_p(T)$ are in $1:1$ correspondence with with irreducible monic polynomials in $\mathbb{F}_p[X]$ and $1/X$. So in the case of $\mathbb{F}_p(X)$ for one gets

$$\operatorname{Spec}\mathbb{F}_p[X] \ni (X-\alpha) \longleftrightarrow \alpha \in \mathbb{F}_p \cup \{\infty\} \longleftrightarrow 1/X$$

Arithmetic $\mathbb{P}^1$

We define points of arithmetic $\mathbb{P}^1$ as prime ideals of $\mathbb{Z}$ corresponding to the non-archimedean norms on $\mathbb{Q}$ compactified by a point at infinity corresponding to the archimedean norm.



Geometric vs arithmetic

Geometric curve - a projective curve over a finite field

Arithmetic curve - prime ideals of a ring of integers of a number field k (number field-a finite extension of $\mathbb{Q}$) with point(s) at ∞ (corresponding to the embeddings of k into $\mathbb{C}$).

Extensions of absolute value functions

As a consequence of Ostrowski's theorem one has a classification of all possible absolute value functions on number fields.

Theorem

Let K be complete with respect to the absolute value function $|\cdot| : K \to \mathbb{R}$ and let L/K be an algebraic extension. Then $|\cdot|$ extends uniquely to an absolute value function on L by the formula

$$|\alpha| = \sqrt[n]{N_{L/K}(\alpha)},$$

where n is the degree $[L : K]$ of the extension.

Here $N_{L/K}$ is the norm function.

Riemann-Roch theorem

Theorem (Riemann-Roch)

For a curve X and a divisor D we have

$$\chi(\mathcal{O}_X(D)) - \chi(\mathcal{O}_X) = \deg D.$$

Riemann-Roch theorem

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For a curve X and a divisor D we have

$$\chi(\mathcal{O}_X(D)) - \chi(\mathcal{O}_X) = \deg D.$$

where χ is the Euler characteristics i.e. the volume of the fundamental domain of the lattice

$$H^0(\mathcal{O}_X(D)) \subset H^0(\mathcal{O}_X(D)) \otimes_{\mathbb{Q}} \mathbb{R}$$

in the arithmetic case and the alternating sum of the dimension of the cohomology groups in the geometric case. In the arithmetic case the divisor D is completed by the archimedean places with coefficients in $\mathbb{R}$.

Unifying approach

Even though the geometric and the arithmetic versions of the Riemann-Roch theorem seem to fundamentally differ from each other there exists a point of view from which they are exactly the same.
This point of view is the Fourier Analysis on global fields (i.e. on Adeles associated to global fields).

Relative Euler characteristic

We will present approach which simultaniously covers the relative case, i.e. the case of a morphism between curves.

It is possible to write the following explicit formula for the Euler characteristic in this case

Definition

For a global field extension L/K one defines the relative Euler characteristic

$$\chi_{L/K}(\mathcal{O}_X(D)) = \chi_L(\mathcal{O}_X(D)) - [L:K]\chi_K(\mathcal{O}_K)$$

Relative Riemann-Roch

Theorem (Relative Riemann-Roch)

Let L/K be an extension of global fields and let D be a divisor on the cirve corresponding to L, then

$$\chi_{L/K}(\mathcal{O}_L(D)) - \chi_{L/K}(\mathcal{O}_L) = \deg D$$

Remark

This explicit theory is compatible with the Grothendieck-Riemann-Roch formalism.

Poisson summation Formula

The Poisson summation formula for adeles $\mathbb{A}_k$ of a number field k has the following form

$$\sum_{\xi \in k} \widehat{f}(\xi) = \sum_{\xi \in k} f(\xi)$$

Theorem (Riemann-Roch)

$$\frac{1}{|\alpha|} \sum_{\xi \in k} \widehat{f}(\xi/\alpha) = \sum_{\xi \in k} f(\alpha\xi)$$

This is just a very straightforward consequence of the Poisson summation formula.

In what follows I will explain what is the ring of adeles of a global field and explain how we can use it to talk uniformly about the Riemann-Roch theory of global fields.

Adeles

We define

$$\mathbb{A}_k = \prod_v{}' k_v$$

where $k = \mathbb{Q}, \mathbb{C}(t)$ and k_v is the topological completion of k with respect to v.

where $\prod'$ -restricted direct product, i.e. (a_v) such that for all but finitely many 'finite' primes a_v are integral (no 'negative' terms in the power series expansion at finite). Equivalently $|a_v|_v < 1$ for all but finitely many 'finite' v.

Remark

Adeles is the minimal subring of the product of all the completions of the global fields such that the diagonal embedding of its field is discrete.

Ideles and Divisors

By ideles we mean the group $\mathbb{A}_K^\times$ of invertible adeles. We have a surjective group homomorphism

$$\begin{array}{ccc} \mathbb{A}_K^\times & \longrightarrow & \mathrm{Div}(X) \\ \Uparrow & & \Uparrow \\ (\alpha)_v & \longmapsto & D_\alpha := \sum_v -n_v(\alpha_v)[v], \end{array}$$

where n_v is the valuation of α_v. In the number field case by $\mathrm{Div}(X)$ we mean the group of divisors completed by the infinite places (where we allow real coefficients).

Let $\alpha = (\alpha_v)_v$ be an idele. We define the idelic absolute value function as:

$$|\alpha| = \prod_v |\alpha_v|_v.$$

Locally compact abelian groups

Let G be a locally compact group, and $\widehat{G}$ its group of characters (continuous homomorphisms into a unit circle), i.e.:

$$\widehat{G} = \{\phi : G \to S^1 \text{ continuous } \}.$$

Then there exist a unique (up to $c > 0$) Haar measure on G. We say that G is self-dual if

$$G \simeq \widehat{G}.$$

The isomorphism is both algebraic and topological.

Self-duality of local fields

For 1-dimensional global field K each of its local fields

K_v is self-dual

in the following way: for a certain chosen character $\psi_0 : K_v \to S^1$ the self duality isomorphism is given by

$$\begin{array}{ccc} K_v & \longrightarrow & \widehat{K_v} \\ \Psi & & \Psi \\ \xi & \longmapsto & x \mapsto \psi_0(\xi x) \end{array}$$

Example

For $K = \mathbb{R}$ one can take

$$\psi_0 : \mathbb{R} \ni \xi \mapsto e^{2\pi i x \xi} \in S^1.$$

Integration on local fields

Lets fix an extension of local fields L_w/K_v.

The Haar μ_w measure on the local field L_w is chosen in the following way:

$$\mu_w(O_w) = N(\mathfrak{d}_{L_w/K_v})^{-1/2}$$

with $v|w$ and

$$\mathfrak{d}_{L_w/K_v} = N_{L_w/K_v}\Big(\{x \in L_w \,|\, \mathrm{Tr}(xO_{L_w}) \subset O_{K_v}\}^{-1}\Big),$$

where $\mathrm{Tr} = \mathrm{Tr}_{L_w/L_v}$ is the trace map.

Remark

The standard measure (as defined in Tate's thesis) is

$$\mu_w(O_w) = N(\mathfrak{d}_{L_w/\mathbb{Q}_p})^{-1/2},$$

where $p|w$ is a prime number.

Local Fourier transform

For a function on L_v we have its Fourier transform

$$\widehat{f}(x) = \int_{L_v} f(\xi)\psi_0(x\xi)d\xi$$

which in the case $K_v = \mathbb{Q}_p$ where $p|w$ is a prime number satisfies the inversion formula

$$\widehat{\widehat{f}}(x) = f(-x).$$

Fourier analysis on adelic spaces

This gives us local compactness, self duality and Fourier analysis for Adeles.

Remark

Used by J.Tate to prove analytic continuation and functional equation of (Hecke) ζ-functions.

We also get that

$$\widehat{K} \simeq \mathbb{A}_K / K.$$

where K is a global field.

Adelic analysis

Adelic functions are functions of the form

$$f(u) = \prod_v f_v(u_v)$$

such that for each $u = (u_v)$ we have that $f_v(u_v) = 1$ for all, but finitely many v.

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We define a Fourier transform

$$\widehat{f}(u) = \int_{\mathbb{A}_K} f(\xi)\psi_0(u\xi)d\xi$$

where ψ_0 is a fixed non-trivial character (product of the characters chosen locally), as the product of the local Fourier transforms.

Elements of the form

$$\alpha = (\alpha_v) \in \mathbb{A} \text{ such that } \alpha_v = 1_v \text{ for almost all } v$$

are in 1 : 1 correspondence with divisors on the curve. For such α we define

$$f_\alpha = \prod_v f_{\alpha_v}(u_v) : \mathbb{A} \to \mathbb{R}$$

where

$$f_{\alpha_v}(u_v) = \begin{cases} \mathrm{char}_{\mathrm{O_v}}(\alpha_v^{-1} u_v) & v \text{ prime} \\ \exp(-\pi|\alpha_v^{-1} u_v|^2) & v \text{ infinite} \end{cases}$$

Fundamental domain of a group action on a set

Recall

Definition

Let G be an abelian group acting on a set X, then the subset $R \subset$ is a fundamental region of X with respect to G if

$$X = \bigcup_{g \in G} g + R$$

and the sum is disjoint.

We will consider $X = \mathbb{A}_L$ and $G = L$. Then

$$R = \prod_{w} O_w \times D$$

where

$$D = \{x = \sum_{i=1}^{n} x_i \omega_i, 0 \leq x_i \leq 1\}$$

and $\omega_1, ...\omega_n$ is the basis of O_L over $\mathbb{Z}$.

Normalizing the Haar measure

For each extension of L/K global fields we can choose the normalization of the Haar measure on the ring of Adeles

$$\mathbb{A}_L$$

in such a way that the volume of the fundamental region (fundamental domain) R of $\mathbb{A}_L/L$ is

$$Vol(R) = Vol(\mathbb{A}_L/L) = d_L^{1/2} \prod_v \prod_{w|v} N(\mathfrak{d}_{L_w/K_v})^{-1/2}.$$

Example

For $K = \mathbb{Q}$ one has

$$Vol(R) = Vol(\mathbb{A}_L/L) = d_L^{1/2} \prod_p \prod_{p|w} N(\mathfrak{d}_{L_w/\mathbb{Q}_p})^{-1/2} = 1.$$

Euler characteristic for curves

We can define

Size of cohomology

$$h^0(D_\alpha) = \log \int_k f_\alpha \qquad \text{with counting measure,}$$

$$h^1(D_\alpha) = \log \int_{\mathbb{A}/k} \bar{f}_\alpha \qquad \text{with probabilistic measure,}$$

where

$$\bar{f}_\alpha : a + k \mapsto \left(\int_k f_\alpha(a+u)du \right) \left(\int_k f_\alpha(u)du \right)^{-1}$$

Using Pontryagin duality and Fourier analysis one can obtain Serre's duality, i.e.

$$h^1(D_\alpha) = h^0(D_{\kappa\alpha^{-1}}).$$

where κ stands for an idele that maps to the canonical divisor through the idele-to-divisor map.

Euler Characteristics

Definition

We define the relative Euler characteristic

$$\chi_{L/K}(D_\alpha) = \log \int_{\mathbb{A}_{L/K}} f_\alpha$$

where by $\int_{\mathbb{A}_{L/K}}$ we mean integration with respect with the normalization of the Haar measure on $\mathbb{A}_L$ corresponding to the extension L/K.

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Notation: By $\int\limits_{\mathbb{A}_L}$ and χ we mean respectively integration and the Euler characteristic with respect to the absolute measure.

Motivation

Thanks to the Weil's equation for the Haar measure we have the following fundamental relation:

$$\int_{\mathbb{A}_K} f_\alpha(u)du = \int_{\mathbb{A}_K/K} \int_K f_\alpha(a+u)du,$$

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and therefore one sees that

$$\int_{\mathbb{A}_K} f_\alpha = \int_K f_\alpha \cdot \int_{\mathbb{A}_K/K} \bar{f}_\alpha.$$

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It is equivalent to

$$\chi(D_\alpha) = h^0(D_\alpha) - h^1(D_\alpha)$$

and justifies the definition of χ.

Calculation of the Euler characteristic

One can easily see that

$$\chi(D_\alpha) = \log \int_{\mathbb{A}} f_\alpha = \log|\alpha| + \log|\kappa|.$$

where $|\kappa|$ is the adelic norm of an idele associated to the canonical divisor i.e.

$$|\kappa|^2 = d_K^{-1} = \begin{cases} q^{2-2g} & \text{in the geometric case} \\ N(\mathfrak{d}_{K/\mathbb{Q}})^{-1} & \text{in the arithmetic case.} \end{cases}$$

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We also have that

$$\chi_{L/K}(D_\alpha) = \chi_L(D_\alpha) - [L:K]\chi_K(D_{1,K})$$

as desired.

Justification

Indeed, the value of the integral for any D_α is uniquely determined by the adelic norm

$$|\cdot| = \prod_w |\cdot|_w$$

and $[\alpha O_{L_w} : O_{L_w}] = \frac{\mu_{L_w/K_v}(\alpha O_{L_w})}{\mu_{L_w/K_v}(O_{L_w})} = |\alpha|_w$ and the local measures $\mu_{L_w/K_v}(O_{L_w})$.

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$$\chi_{L/K}(D_\alpha) = \log|\alpha| + \log|\kappa_{L/K}|$$

where $\kappa_{L/K}$ is the idele associated to the divisor of the relative discriminant $\mathfrak{d}_{L/K}$.

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$$\log|\kappa_{L/K}| = \log|\kappa_L| - [L:K]\log|\kappa_K|$$

cause

$$\prod_v \prod_{w|v} N(\mathfrak{d}_{L_w/K_v})^{-1} = d_K^{[L:K]}/d_L \qquad \mathfrak{d}_{L/K} = \prod_v \prod_{w|v} \mathfrak{d}_{L_w/K_v}.$$