Introduction to Diophantine approximation and a generalisation of Roth's theorem

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Joint with Francesco Zucconi

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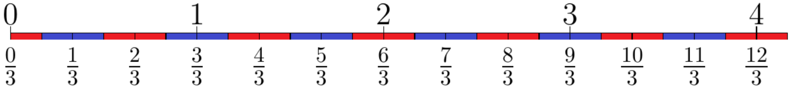

Figure: Spacing with s=3

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$$\tau(\xi) := \sup \left\{ t \in \mathbb{R} \colon 0 < \left| \xi - \frac{r}{s} \right| < |s|^{-t} \ \begin{array}{c}\text{has infinitely}\\ \text{many coprime}\\ \text{solutions } (r,s) \in \mathbb{Z}^2\end{array} \right\}$$

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Roughly speaking the Definition says that ξ can be approximated in an "unlucky" way up to the order $\tau(\xi)$ so in particular any other "better" approximation of ξ is indeed "lucky".

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Theorem (Dirichlet, 1840)

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The proof is a nice application of the pigeonhole principle.

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- Gelfond-Dyson: $\tau(\xi) \leq \sqrt{2d}$

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An equivalent formulation of the theorem is the following:

Theorem (Roth)

Let $\xi \in \mathbb{R}$ be an algebraic number, and let $\varepsilon > 0$ be a real number. Then there exists a real constant $C(\xi, \varepsilon) > 0$ such that for every pair of coprime integers (r, s) with $s > C(\xi, \varepsilon)$, it holds that:

$$\left|\xi - \frac{r}{s}\right| > |s|^{-(2+\varepsilon)} \tag{1.1}$$

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- For any $\lambda \in [2, +\infty[$, there exists $\xi \in \mathbb{R}$ such that $\tau(\xi) = \lambda$.
- For almost all $\xi \in \mathbb{R}$ in the sense of Lebesgue we have that $\tau(\xi) = 2$.

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- Simultaneous approximation of n elements algebraic over k.

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$$\begin{aligned} \phi\colon \Omega &\to M_{\mathbb{K}} \\ \omega &\mapsto |\cdot|_\omega := \phi(\omega)\,. \end{aligned}$$

such that for any $a \in \mathbb{K}^\times$, the function $\omega \mapsto \log|a|_\omega$ lies in $L^1(\Omega, \mu)$.

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such that for any $a \in \mathbb{K}^\times$, the function $\omega \mapsto \log|a|_\omega$ lies in $L^1(\Omega, \mu)$. The triple $\mathbb{X} = (\mathbb{K}, \Omega, \phi)$ is called an adelic curve.

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An adelic curve $\mathbb{X} = (\mathbb{K}, \Omega, \phi)$ is said to be proper if for any $a \in \mathbb{K}^{\times}$:

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where ν denotes a generic element of $\overline{\Omega}$ and χ is the measure on $\overline{\Omega}$.

Examples of proper adelic curves

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- Function fields of polarized arithmetic varieties. (Vojta's case).

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- μ-equicontinuity
- uniform integrability

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Theorem

Let $\mathbb{X} = (\mathbb{K}, \Omega, \phi)$ be a proper adelic curve satisfying the μ-equicontinuity condition.

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Playing a bit with the theory of extensions of absolute values one can show that this theorem implies Corvaja's result.

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$$\int_S \min_{1 \le i \le n} \left(\log^- |\beta - \alpha_i|_\omega \right) d\mu(\omega) > -(2+\varepsilon) h_{\mathbb{X}}(\beta) + c \qquad (2.2)$$

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- In Corvaja's setting the validity of the technical hypotheses is trivial, since we have the counting measure.
- $\overline{\mathbb{Q}}$ is an adelic curve but it is not hard to show that Roth's theorem fails.

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The existence of an "interpolating polynomial" δ for $\alpha_1, \ldots, \alpha_n$ allows to write some integral bounds for measurable functions on $\theta : \Omega \to \mathbb{R}_{\geq 0}$ satisfying some technical properties with respect to heights of the α_i's and β.

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- (Hard). A version of the Schmidt subspace theorem for adelic curves. Here the obstruction is the absence of Minkowski's theory.

Thank You