On pointwise convergence problems, part III

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Bourgain's pointwise ergodic theorem

In the early 1980's Bellow and Furstenberg asked independently about the poinwise convergence for polynomial ergodic averages

$$A^P_{N;X,T}f(x) := \frac{1}{N}\sum_{n=1}^{N} f(T^{P(n)}x) \quad \text{for} \quad x \in X,$$

where $P \in \mathbb{Z}[\mathrm{n}]$ is a polynomial of degree > 1.

Bourgain used the circle method of Hardy and Littlewood to show:

- $L^p(X)$ boundedness of the maximal function for any $1 < p \le \infty$, i.e.

$$\| \sup_{N\in\mathbb{N}} |A^P_{N;X,T}f| \|_{L^p(X)} \lesssim \|f\|_{L^p(X)} \quad \text{for} \quad p \in (1,\infty].$$

- Given an increasing sequence $(N_j : j \in \mathbb{N})$, for each $J \in \mathbb{N}$ one has

$$\Big(\sum_{j=0}^{J} \big\| \sup_{N_j \le N < N_{j+1}} \big|A^P_{N;X,T}f - A^P_{N_j;X,T}f\big| \big\|^2_{L^2(X)}\Big)^{1/2} \le o(J^{1/2})\|f\|_{L^2(X)}.$$

Bourgain's maximal ergodic theorem for $M_N^P = A_{N;\mathbb{Z},S}^P$

- We prove that
$$\big\| \sup_{n\in\mathbb{N}} |M_{2^n}^P f| \big\|_{\ell^2(\mathbb{Z})} \lesssim \|f\|_{\ell^2(\mathbb{Z})}.$$
To simplify arguments assume that $P(x) = x^d$ and $d \geq 2$.
- Let
$$K_N(x) = \frac{1}{N}\sum_{k=1}^{N} \delta_{P(k)}(x),$$
then
$$M_N^P f(x) = K_N * f(x).$$
- For $f \in \ell^1(\mathbb{Z})$ let
$$\widehat{f}(\xi) = \sum_{k\in\mathbb{Z}} e^{2\pi i \xi k} f(k)$$
and observe that
$$m_N(\xi) = \widehat{K}_N(\xi) = \frac{1}{N}\sum_{k=1}^{N} e^{2\pi i \xi k^d} \quad (\xi \in \mathbb{T}).$$
- Consequently
$$M_N^P f(x) = K_N * f(x) = \mathcal{F}^{-1}\big(m_N \hat{f}\big)(x).$$

Some heuristics

- First of all we have to understand the behaviour of

$$m_N(\xi) = \frac{1}{N}\sum_{k=1}^{N} e^{2\pi i \xi k^d},$$

  and we would like to replace $m_N(\xi)$ with the integral

$$\Phi_N(\xi) = \int_0^1 e^{2\pi i \xi (Nx)^d} dx.$$

- We can not do this naively, since the derivative of the phase function $k^d\xi$ arising in the exponential sum is equal to $dk^{d-1}\xi$ and may be large.
- In general we have no control over the error term

$$m_N(\xi) - \Phi_N(\xi).$$

Gaussian sums

- If $\xi = a/q$ and $(a,q)=1$ then we see that $m_N(a/q)$ behaves like a complete Gaussian sum

$$G(a/q) = \frac{1}{q}\sum_{r=1}^{q} e^{2\pi i \frac{a}{q} r^d}.$$

- Indeed,

$$m_N(a/q) = \frac{1}{N}\sum_{k=1}^{N} e^{2\pi i \frac{a}{q} k^d} = \frac{1}{N}\sum_{r=1}^{q}\sum_{-\frac{r}{q}<k\leq\frac{N-r}{q}} e^{2\pi i \frac{a}{q}(qk+r)^d} \simeq \frac{1}{q}\sum_{r=1}^{q} e^{2\pi i \frac{a}{q} r^d}.$$

- This suggests that the asymptotics for m_N should be concentrated in some neighbourhoods of Diophantine approximations of ξ with small denominators.

Small denominators - asymptotic formula for $m_N(\xi)$

- From Dirichlet's principle for any $\xi \in [0,1]$ and we can always find $a/q \in [0,1)$ such that $1 \le q \le N^{d-\beta}$, $(a,q)=1$ and

$$\left|\xi - \frac{a}{q}\right| \le \frac{1}{qN^{d-\beta}}$$

for any $\beta > 0$. If $1 \le q \le N^{\beta}$ then

$$m_N(\xi) = \frac{1}{N}\sum_{k=1}^{N} e^{2\pi i\xi\cdot k^d} = \frac{1}{N}\sum_{r=1}^{q}\sum_{-\frac{r}{q}<n\le\frac{N-r}{q}} e^{2\pi i(\xi-\frac{a}{q})(qn+r)^d} e^{2\pi i\frac{a}{q}(qn+r)^d}$$

$$= \frac{1}{qN}\sum_{r=1}^{q} e^{2\pi i\frac{a}{q}r^d}\cdot\frac{q}{N}\sum_{-\frac{r}{q}<n\le\frac{N-r}{q}} e^{2\pi i(\xi-\frac{a}{q})(qn+r)^d}$$

$$= \left(\frac{1}{q}\sum_{r=1}^{q} e^{2\pi i\frac{a}{q}r^d}\right)\cdot\left(\int_0^1 e^{2\pi i(\xi-\frac{a}{q})(Nx)^d}dx\right) + \mathcal{O}(N^{-1/2}).$$

- Therefore, if ξ is in the neighbourhood of a/q as above, we have

$$m_N(\xi) = G(a/q)\cdot\Phi_N(\xi - a/q) + \mathcal{O}(N^{-1/2}).$$

Large denominators - Weyl's inequality

- It was observed by Hardy and Littlewood that if $|\xi - a/q| \le \frac{1}{qN^{d-\beta}} \le q^{-2}$ and $(a,q) = 1$ and $N^{\beta} \le q \le N^{d-\beta}$ then

$$|m_N(\xi)| = \left|\frac{1}{N}\sum_{k=1}^{N} e^{2\pi i \xi k^d}\right| \lesssim N^{-\alpha}$$

  for some $\alpha \in (0,1)$. This follows from Weyl's inequality.

Lemma (Weyl's inequality)

Let $P(x) = a_d x^d + \ldots + a_1 x$. Suppose there are $(a,q) = 1$ such that $|a_d - a/q| \le q^{-2}$. Then there is $C > 0$ such that

$$\frac{1}{N}\left|\sum_{m=1}^{N} e^{2\pi i P(m)}\right| \le C(\log N)^2 \left(\frac{1}{q} + \frac{1}{N} + \frac{q}{N^d}\right)^{1/2^{d-1}}$$

uniformly in N and q.

- Observe also that for some $\delta > 0$ we have

$$|G(a/q)| = \left|\frac{1}{q}\sum_{r=1}^{q} e^{2\pi i \frac{a}{q} r^d}\right| \lesssim q^{-\delta}.$$

Projections $\Xi_{2^n}(\xi)$

- For $\varepsilon, \chi > 0$ let us define the following projections

$$\Xi_N(\xi) = \sum_{a/q \in \mathscr{R}_{\leq N^\varepsilon}} \eta(N^{(d-\chi)}(\xi - a/q))$$

  with a smooth cuf-off function η and

$$\mathscr{R}_{\leq N} = \{a/q \in \mathbb{T} : \ (a,q) = 1 \text{ and } 1 \leq q \leq N\}.$$

- Since

$$m_{2^n}(\xi) = m_{2^n}(\xi)(1 - \Xi_{2^n}(\xi)) + m_{2^n}(\xi)\Xi_{2^n}(\xi),$$

  the first term is highly oscillatory, as supported in the regime where Weyl's inequality is efficient.
- The second term provides asymptotic and will be approximated by the integral

$$\Phi_N(\xi) = \int_0^1 e^{2\pi i \xi (Nx)^d} dx.$$

Highly oscillatory part: $m_{2^n}(1-\Xi_{2^n})$

- From Weyl's inequality we have

$$|m_{2^n}(\xi)| = \Big|\frac{1}{2^n}\sum_{k=1}^{2^n} e^{2\pi i \xi k^d}\Big| \lesssim 2^{-\alpha n}$$

  for a large $\alpha > 0$, provided that $1-\Xi_{2^n}(\xi) \neq 0$.

- Therefore, by Plancherel's theorem

$$\begin{aligned}\big\|\sup_{n\in\mathbb{N}}|\mathcal{F}^{-1}\big(m_{2^n}(1-\Xi_{2^n})\hat{f}\big)|\big\|_{\ell^2} &\leq \sum_{n\in\mathbb{N}_0}\big\|\mathcal{F}^{-1}\big(m_{2^n}(1-\Xi_{2^n})\hat{f}\big)\big\|_{\ell^2}\\ &\lesssim \sum_{n\in\mathbb{N}_0} 2^{-\alpha n}\|f\|_{\ell^2}\\ &\lesssim \|f\|_{\ell^2}.\end{aligned}$$

Asymptotic part: $m_{2^n}\Xi_{2^n}$

- Recall that if $a/q \in \mathscr{R}_{\leq 2^{\varepsilon n}}$ then we have

$$m_{2^n}(\xi) \simeq G(a/q)\cdot\Phi_{2^n}(\xi-a/q) = \Big(\frac{1}{q}\sum_{r=1}^{q} e^{2\pi i \frac{a}{q} r^d}\Big)\cdot\Big(\int_0^1 e^{2\pi i(\xi-\frac{a}{q})(2^n x)^d}dx\Big).$$

- Therefore,

$$m_{2^n}(\xi)\Xi_{2^n}(\xi) \simeq \sum_{s\geq 0} m_{2^n}^s(\xi)$$

where

$$m_{2^n}^s(\xi) = \sum_{a/q\in\mathscr{R}_{2^s}} G(a/q)\Phi_{2^n}(\xi-a/q)\eta\big(2^{s(d-\chi)}(\xi-a/q)\big),$$

with $\mathscr{R}_{2^s} = \{a/q \in \mathbb{T}:\ (a,q)=1 \text{ and } 2^{s-1} < q \leq 2^s\}$.

- The task now is to show that for any $s \geq 0$ we have

$$\big\|\sup_{n\in\mathbb{N}} |\mathcal{F}^{-1}\big(m_{2^n}^s\hat{f}\big)|\big\|_{\ell^2} \lesssim 2^{-\delta s}\|f\|_{\ell^2}, \qquad f\in\ell^2(\mathbb{Z}),$$

where $\delta > 0$ comes from the estimate $|G(a/q)| \lesssim q^{-\delta}$.

The case $n \geq 2^{\kappa s}$

- We split the supremum into two parts $0 \leq n \leq 2^{\kappa s}$ and $n \geq 2^{\kappa s}$ for some $\kappa \in \mathbb{N}$ to be specified later.

$$\Big\| \sup_{n \in \mathbb{N}} |\mathcal{F}^{-1}\big(m^s_{2^n}\hat{f}\big)| \Big\|_{\ell^2} \leq \Big\| \sup_{n \geq 2^{\kappa s}} |\mathcal{F}^{-1}\big(m^s_{2^n}\hat{f}\big)| \Big\|_{\ell^2} + \Big\| \sup_{0 \leq n \leq 2^{\kappa s}} |\mathcal{F}^{-1}\big(m^s_{2^n}\hat{f}\big)| \Big\|_{\ell^2}.$$

- For the first term we show that

$$\Big\| \sup_{n \geq 2^{\kappa s}} |\mathcal{F}^{-1}\big(m^s_{2^n}\hat{f}\big)| \Big\|_{\ell^2} \lesssim 2^{-\delta s} \sup_{\|g\|_{L^2(\mathbb{R})}=1} \Big\| \sup_{R>0} \Big| R^{-1} \int_0^R g(x - t^d)\,dt \Big| \Big\|_{L^2(\mathbb{R})} \|f\|_{\ell^2(\mathbb{Z})},$$

which by the Hardy–Littlewood maximal theorem for $p \in (1, \infty)$ one can conclude that

$$\Big\| \sup_{R>0} \Big| R^{-1} \int_0^R g(x - t^d)\,dt \Big| \Big\|_{L^p(\mathbb{R})} \lesssim \|g\|_{L^p(\mathbb{R})}.$$

The case $0 \leq n \leq 2^{\kappa s}$

Rademacher–Menshov inequality

For any sequence $(a_j)_{0\leq j\leq 2^s} \subseteq \mathbb{C}$ and any $s \in \mathbb{N}$ we have

$$\sup_{0\leq n\leq 2^s} |a_n| \leq |a_0| + \sqrt{2}\sum_{i=0}^{s}\Big(\sum_{j=0}^{2^{s-i}-1} |a_{(j+1)2^i} - a_{j2^i}|^2\Big)^{1/2}$$

- Hence by Plancherel's theorem we obtain

$$\Big\|\sup_{0\leq n\leq 2^{\kappa s}} |\mathcal{F}^{-1}(m^s_{2^n}\hat{f})|\Big\|_{\ell^2}$$

$$\lesssim \Big\|\sum_{i=0}^{\kappa s}\Big(\sum_{j=0}^{2^{\kappa s-i}-1}\Big(\sum_{k=j2^i}^{(j+1)2^i-1}\mathcal{F}^{-1}\big((m^s_{2^{k+1}} - m^s_{2^k})\hat{f}\big)\Big)^2\Big)^{1/2}\Big\|_{\ell^2}$$

$$\lesssim \sum_{i=0}^{\kappa s}\Big(\sum_{j=0}^{2^{\kappa s-i}-1}\Big\|\sum_{k=j2^i}^{(j+1)2^i-1}(m^s_{2^{k+1}} - m^s_{2^k})\hat{f}\Big\|_{L^2}^2\Big)^{1/2}.$$

The case $0 \le n \le 2^{\kappa s}$

$$\Big\| \sup_{0\le n\le 2^{\kappa s}} |\mathcal{F}^{-1}(m_{2^n}\hat{f})| \Big\|_{\ell^2} \lesssim \sum_{i=0}^{\kappa s} \Bigg(\sum_{j=0}^{2^{\kappa s-i}-1} \Big\| \sum_{k=j2^i}^{(j+1)2^i-1} (m^s_{2^{k+1}} - m^s_{2^k})\hat{f} \Big\|_{L^2}^2 \Bigg)^{1/2}$$

$$\lesssim \sum_{l=0}^{\kappa s} \Bigg(\sum_{j=0}^{2^{\kappa s-l}-1} \sum_{j2^l\le k,k'<(j+1)2^l} \int_{\mathbb{T}} |m^s_{2^{k+1}}(\xi) - m^s_{2^k}(\xi)||m^s_{2^{k'+1}}(\xi) - m^s_{2^{k'}}(\xi)||\hat{f}(\xi)|^2 d\xi \Bigg)^{1/2}$$

$$\lesssim s\Bigg(\sum_{a/q\in\mathscr{R}_{2^s}} |G(a/q)|^2 \int_{\mathbb{T}} |\hat{f}(\xi)|^2 \eta\big(2^{s(d-\chi)}(\xi - a/q)\big)^2 d\xi \Bigg)^{1/2}$$

since

$$\sum_{j\in\mathbb{Z}} |\Phi_{2^{j+1}}(\xi) - \Phi_{2^j}(\xi)| \lesssim \sum_{j\in\mathbb{Z}} \min\left\{|2^j\xi|, |2^j\xi|^{-1/d}\right\} \lesssim 1.$$

Finally we obtain

$$\Big\| \sup_{0\le n\le 2^{\kappa s}} |\mathcal{F}^{-1}(m^s_{2^n}\hat{f})| \Big\|_{\ell^2} \lesssim s2^{-\delta s}\|f\|_{\ell^2}.$$

L^p good and bad sequences

Theorem (Bourgain's ergodic theorem, (1989))

Let $(X, \mathcal{B}(X), \mu, T)$ be a σ-finite measure preserving system. For every $1 < p < \infty$ and $P \in \mathbb{Z}[n]$ and $f \in L^p(X)$ there exists $f^* \in L^p(X)$ such that

$$\frac{1}{N}\sum_{n=1}^{N} f(T^{P(n)}x) \xrightarrow[N\to\infty]{} f^*(x).$$

- This theorem gives a positive answer to Bellow and Furstenberg problem, and inspired many authors to investigate the averages

$$A_N^{a_n} f(x) = \frac{1}{N}\sum_{n=1}^{N} f(T^{a_n}x)$$

defined along sequences $(a_n)_{n\in\mathbb{N}} \subset \mathbb{Z}$. We will say that

- $(a_n)_{n\in\mathbb{N}}$ is L^p-good if the pointwise convergence of $A_N^{a_n} f(x)$ holds for every dynamical system $(X, \mathcal{B}(X), \mu, T)$ and every $f \in L^p(X)$.
- Otherwise $(a_n)_{n\in\mathbb{N}}$ is L^p-bad.

Examples

- $a_n = n$ is L^p-good for $p \geq 1$,
  – Birkhoff ergodic theorem (1931).
- $a_n = P(n)$ is L^p-good for $p > 1$, where $P \in \mathbb{Z}[n]$,
  – Bourgain (1989).
- $a_n =$ n-th prime number, is L^p-good for $p > 1$,
  – Bourgain/Wierdl (1989).

The question about the endpoint estimates $p = 1$ was a major open problem in pointwise ergodic theory that led to a remarkable conjecture:

Conjecture (Rosenblatt & Wierdl (1991))

There are no sequences $(a_n)_{n\in\mathbb{N}} \subseteq \mathbb{N}$, with gaps tending to infinity, i.e.

$$\lim_{n\to\infty} (a_{n+1} - a_n) = \infty,$$

which are L^1-good.

Examples

Meanwhile, L^p theory (for $p > 1$) has been developed, and it was shown

- $a_n = \lfloor h(n) \rfloor$ is L^p-good for $p > 1$,
  – Boshernitzan, Kolesnik, Quas & Wierdl (2005), where

$$h(x) = x^c \log^A x,$$
$$h(x) = x^c e^{A \log^B x},$$
$$h(x) = x^c \log\log \dots \log x,$$

  with $c \geq 1$, $A \in \mathbb{R}$, $B \in (0,1)$.

- $a_n = \lfloor n^c \rfloor$ is L^1-good for $1 < c < 1.001$,
  – Urban & Zienkiewicz (2007).
  This gives a negative answer to Rosenblatt and Wierdl's conjecture.

- $a_n = n^k$ is L^1-bad
  – for $k = 2$, Buczolich & Mauldin (2010),
  – for $k \geq 2$, LaVictoire (2011).

- $a_n = n$-th prime number, is L^1-bad
  – LaVictoire (2011).

Other L^1 good sequences

Theorem (M. (2013))

Assume that $1 < c < 30/29 \simeq 1.0345$, *and let h be a function of the form*

$$h(n) = n^c\, L(n),$$

where $L(n)$ *is a slowly varying function (satisfying certain smoothness conditions). Then the sequence*

$$a_n = \lfloor h(n) \rfloor \quad \text{is } L^1\text{-good.}$$

In particular, there is a constant $C > 0$ *such that for any* σ*-finite measure preserving system* $(X, \mathcal{B}(X), \mu, T)$ *we have*

$$\mu\Big(\Big\{x \in X\colon \sup_{N\in\mathbb{N}} \Big|\frac{1}{N}\sum_{n=1}^{N} f(T^{a_n}x)\Big| > \lambda\Big\}\Big) < \frac{C}{\lambda}\|f\|_{L^1(X)}$$

for every $\lambda > 0$ *and* $f \in L^1(X)$.

Bergelson–Richter prime number theorem

Let $\Omega(n)$ denote the number of prime factors of a natural number $n \in \mathbb{N}$ counted with multiplicities.

Theorem (Bergelson–Richter theorem (2020))

Let (X, μ, T) be uniquely ergodic topological system. Then

$$\frac{1}{N}\sum_{k=1}^{N} f(T^{\Omega(k)}x) \xrightarrow[N\to\infty]{} \int_X f(y)d\mu(y),$$

for every every $x \in X$ and $f \in C(X)$. In particular, for every $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ the sequence $(\{\alpha\Omega(n)\})_{n\in\mathbb{N}}$ is equidistributed.

Surprisingly, unique ergodicity and continuity are essential, as we have that

Theorem (Loyd's theorem (2023))

For any non-atomic ergodic probability measure preserving system $(X, \mathcal{B}(X), \mu, T)$ there exists a measurable set $A \in \mathcal{B}(X)$ such that

$$\liminf_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} \mathbb{1}_A(T^{\Omega(k)}x) = 0 \quad \textit{and} \quad \limsup_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} \mathbb{1}_A(T^{\Omega(k)}x) = 1,$$

for almost all $x \in X$.

Oscillation inequality for Birkhoff's operators

Recall that

$$A_{N;X,T}f(x) = \frac{1}{N}\sum_{n=1}^{N} f(T^n x).$$

Fix $\tau \in (1,2]$ and define $\Lambda = \{\lfloor \tau^k \rfloor : k \in \mathbb{N} \cup \{0\}\}$. Let $(k_j)_{j\in\mathbb{N}}$ be an increasing sequence and set $N_j = \lfloor \tau^{k_j} \rfloor$.

Theorem

Let $(X, \mathcal{B}(X), \mu, T)$ be a measure-preserving system then for every $J \in \mathbb{N}$ there is $C_J > 0$ such that we have

$$\sum_{j=0}^{J} \Big\| \sup_{N \in \Lambda \cap (N_j, N_{j+1}]} \big| A_{N;X,T}f - A_{N_j;X,T}f \big| \Big\|_{L^2(X)}^2 \le C_J \|f\|_{L^2(X)}^2, \qquad (1)$$

and $\lim_{J\to\infty} C_J/J = 0$. In particular, for every $f \in L^2(X)$ there exists $f^* \in L^2(X)$ such that

$$\lim_{N\to\infty} A_{N;X,T}f(x) = f^*(x),$$

for μ-almost every $x \in X$.

Proof of the oscillation inequality for Birkhoff's operators

- Repeating the same argument as in the proof of transference principle it only suffices to work with $(\mathbb{Z}, \mathbf{P}(\mathbb{Z}), |\cdot|, S)$ with $S(x) = x - 1$. Then

$$A_{N;X,S} f(x) = M_N f(x) = \frac{1}{N} \sum_{n=1}^{N} f(x-n) = K_N * f(x), \qquad f \in \ell^2(\mathbb{Z}),$$

where

$$K_N(x) = \frac{1}{N} \sum_{n=1}^{N} \delta_n(x), \qquad x \in \mathbb{Z}.$$

- By the bounds for the Hardy–Littlewood maximal function

$$\| \sup_{N \in \mathbb{N}} |M_N f| \|_{\ell^p(\mathbb{Z})} \lesssim \|f\|_{\ell^p(\mathbb{Z})}$$

one can assume that $f \in \ell^2(\mathbb{Z}) \cap \ell^\infty(\mathbb{Z})$ and $f \geq 0$ is finitely supported.
- For $f \in \ell^1(\mathbb{Z})$ let us denote by

$$\hat{f}(\xi) = \sum_{n \in \mathbb{Z}} e^{2\pi i n \xi} f(n),$$

the discrete Fourier transform on $\mathbb{Z}$ and let $\mathcal{F}^{-1}$ be its inverse.

Proof of the oscillation inequality for Birkhoff's operators

- One can see that $\widehat{M_N f}(\xi) = \hat{K}_N(\xi)\hat{f}(\xi)$, where

$$\hat{K}_N(\xi) = \frac{1}{N}\sum_{n=1}^{N} e^{2\pi i n\xi}.$$

- Let $B_j = \{x \in (-1/2, 1/2) : |x| \le N_j^{-1}\}$. By Plancherel's theorem

$$\begin{aligned}
&\sum_{j=0}^{J} \Big\| \sup_{N\in\Lambda\cap(N_j,N_{j+1}]} \big|\mathcal{F}^{-1}\big((\hat{K}_N - \hat{K}_{N_j})\mathbb{1}_{B_{j+1}}\hat{f}\big)\big|\Big\|_{\ell^2}^2 \\
&\qquad \le \sum_{j=0}^{J} \sum_{N\in\Lambda\cap(N_j,N_{j+1}]} \big\|\mathcal{F}^{-1}\big((\hat{K}_N - \hat{K}_{N_j})\mathbb{1}_{B_{j+1}}\hat{f}\big)\big\|_{\ell^2}^2 \\
&\qquad\qquad \le \Big\| \sum_{j=0}^{J} \mathbb{1}_{B_{j+1}} \sum_{N\in\Lambda\cap(N_j,N_{j+1}]} |\hat{K}_N - \hat{K}_{N_j}|^2 \Big\|_{L^\infty} \|f\|_{\ell^2}^2.
\end{aligned}$$

Proof of the oscillation inequality for Birkhoff's operators

- For $N \in \Lambda \cap (N_j, N_{j+1}]$ we have

$$|\hat{K}_N(\xi) - \hat{K}_{N_j}(\xi)| \lesssim |\xi| N,$$

hence

$$\sum_{j=0}^{J} \mathbb{1}_{B_{j+1}}(\xi) \sum_{N \in \Lambda \cap (N_j, N_{j+1}]} |\hat{K}_N(\xi) - \hat{K}_{N_j}(\xi)|^2$$

$$\lesssim |\xi|^2 \sum_{j=0}^{J} \mathbb{1}_{B_{j+1}}(\xi) \sum_{N \in \Lambda \cap (N_j, N_{j+1}]} N^2$$

$$\lesssim |\xi|^2 \sum_{j : N_{j+1} \leq |\xi|^{-1}} N_{j+1}^2 \lesssim 1.$$

Therefore, we obtain

$$\sum_{j=0}^{J} \Big\| \sup_{N \in \Lambda \cap (N_j, N_{j+1}]} \big| \mathcal{F}^{-1}\big((\hat{K}_N - \hat{K}_{N_j}) \mathbb{1}_{B_{j+1}} \hat{f} \big) \big| \Big\|_{\ell^2}^2 \lesssim \|f\|_{\ell^2}^2.$$

Proof of the oscillation inequality for Birkhoff's operators

- Similar for B_j^c, replacing $\hat{K}_{N_j}$ by $\hat{K}_{N_{j+1}}$ under the supremum, we can estimate

$$\sum_{j=0}^{J} \Big\| \sup_{N\in\Lambda\cap(N_j,N_{j+1}]} \big|\mathcal{F}^{-1}\big((\hat{K}_N - \hat{K}_{N_j})\mathbb{1}_{B_j^c}\hat{f}\big)\big|\Big\|_{\ell^2}^2$$

$$\lesssim \sum_{j=0}^{J} \sum_{N\in\Lambda\cap[N_j,N_{j+1}]} \big\|\mathcal{F}^{-1}\big((\hat{K}_{N_{j+1}} - \hat{K}_N)\mathbb{1}_{B_j^c}\hat{f}\big)\big\|_{\ell^2}^2$$

$$\leq \Big\| \sum_{j=0}^{J} \mathbb{1}_{B_j^c} \sum_{N\in\Lambda\cap[N_j,N_{j+1}]} |\hat{K}_{N_{j+1}} - \hat{K}_N|^2 \Big\|_{L^\infty} \|f\|_{\ell^2}^2.$$

Now for $N \in \Lambda \cap [N_j, N_{j+1}]$ we obtain

$$|\hat{K}_{N_{j+1}}(\xi) - \hat{K}_N(\xi)| \lesssim |\xi|^{-1} N^{-1}$$

Proof of the oscillation inequality for Birkhoff's operators

- Thus

$$\sum_{j=0}^{J} \mathbb{1}_{B_j^c}(\xi) \sum_{N\in\Lambda\cap[N_j,N_{j+1}]} |\hat{K}_{N_{j+1}}(\xi) - \hat{K}_N(\xi)|^2$$

$$\lesssim |\xi|^{-2} \sum_{j=0}^{J} \mathbb{1}_{B_j^c}(\xi) \sum_{N\in\Lambda\cap[N_j,N_{j+1}]} N^{-2}$$

$$\lesssim |\xi|^{-2} \sum_{j:N_j\geq|\xi|^{-1}} N_j^{-2} \lesssim 1.$$

Therefore, we conclude

$$\sum_{j=0}^{J} \Big\| \sup_{N\in\Lambda\cap(N_j,N_{j+1}]} \big|\mathcal{F}^{-1}\big((\hat{K}_N - \hat{K}_{N_j})\mathbb{1}_{B_j^c}\hat{f}\big)\big|\Big\|_{\ell^2}^2 \lesssim \|f\|_{\ell^2}^2.$$

Proof of the oscillation inequality for Birkhoff's operators

- Finally, for $p = 2$ we obtain

$$\sum_{j=0}^{J} \Big\| \sup_{N \in \Lambda \cap (N_j, N_{j+1}]} \big| \mathcal{F}^{-1}\big((\hat{K}_N - \hat{K}_{N_j}) \mathbb{1}_{B_j} \mathbb{1}_{B_{j+1}^c} \hat{f} \big) \big| \Big\|_{\ell^2}^2 \lesssim \sum_{j=0}^{J} \big\| \mathcal{F}^{-1}\big(\mathbb{1}_{B_j} \mathbb{1}_{B_{j+1}^c} \hat{f} \big) \big\|_{\ell^2}^2 \leq \|f\|_{\ell^2}^2.$$

- Hence

$$\sum_{j=0}^{J} \Big\| \sup_{N \in \Lambda \cap (N_j, N_{j+1}]} \big| A_{N;X,T} f - A_{N_j;X,T} f \big| \Big\|_{L^2(X)}^2 \leq C_J \|f\|_{L^2(X)}^2,$$

- In fact we have proved that C_J is constant. The proof of (1) is completed.

How oscillations imply pointwise convergence

- By the maximal inequality for $p = 2$ we can assume that $f \in L^2(X)$ is bounded and $\|f\|_{L^\infty(X)} \leq 1$.
- Suppose for a contradiction that $(A_{N;X,T}f(x))_{N\in\mathbb{N}}$ does not converge. Then there is $\varepsilon \in (0, 1)$ such that

$$\mu\Big(\{x \in X : \limsup_{M,N\to\infty} |A_{M;X,T}f(x) - A_{N;X,T}f(x)| > 8\varepsilon\}\Big) > 8\varepsilon.$$

- Thus there exists $(k_j)_{j\in\mathbb{N}}$ such that

$$\mu\Big(\{x \in X : \sup_{N_j<N\leq N_{j+1}} |A_{N;X,T}f(x) - A_{N_j;X,T}f(x)| > 4\varepsilon\}\Big) > 4\varepsilon,$$

  where $N_j = \lfloor \tau^{k_j} \rfloor$ and $\tau = 1 + \varepsilon/4$.
- If $\lfloor \tau^k \rfloor \leq N < \lfloor \tau^{k+1} \rfloor$ then

$$|A_{N;X,T}f - A_{\lfloor \tau^k \rfloor;X,T}f| = \left| \frac{1}{N} \sum_{n=\lfloor \tau^k \rfloor+1}^{N} f(T^n x) - \frac{N - \lfloor \tau^k \rfloor}{N\lfloor \tau^k \rfloor} \sum_{n=1}^{\lfloor \tau^k \rfloor} f(T^n x) \right|$$

$$\leq \frac{2(N - \lfloor \tau^k \rfloor)}{N} \leq \frac{4\tau^k(\tau - 1)}{\tau^k} + \frac{4}{\tau^k} = 4(\tau - 1) + \frac{4}{\tau^k} < 2\varepsilon,$$

  for $k \geq k_0$, since we can always arrange k_0 to satisfy $\tau^{-k_0} < \varepsilon/4$.

How oscillations imply pointwise convergence

- Therefore, we obtain that

$$\mu\Big(\{x \in X : \sup_{N \in \Lambda \cap (N_j, N_{j+1}]} |A_{N;X,T}f(x) - A_{N_j;X,T}f(x)| > \varepsilon\}\Big) > \varepsilon.$$

- Now applying oscillation inequality we obtain that

$$0 < \varepsilon^3 \leq \frac{1}{J} \sum_{j=0}^{J} \Big\| \sup_{N \in \Lambda \cap (N_j, N_{j+1}]} \big|A_{N;X,T}f - A_{N_j;X,T}f\big| \Big\|_{L^2(X)}^2 \leq J^{-1} C_J \|f\|_{L^2(X)}^2,$$

  but it is impossible since, the right-hand side tends to 0 as $J \to \infty$.
- This proves the pointwise convergence of $A_{N;X,T}f$ on $L^2(X)$ and completes the proof.

Dziękuję!