Ranks of universal quadratic forms over quadratic fields

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Quadratic form: $Q(x_1, \dots, x_r) = \sum_{1 \le i \le j \le r} a_{ij} x_i x_j, \quad a_{ij} \in \mathbb{Z}$.

r is the rank of Q.

Which integers are represented by Q?

Rich history:
(1) Pythagorean triples: $x^2 + y^2 - z^2 = 0$ (Babylonia $\sim$ 1800BC),
(2) Pell equation: $x^2 - dy^2 = 0$ (India $\sim$ 600BC, Fermat $\sim$ 1650s),
(3) Sum of four squares: $x^2 + y^2 + z^2 + t^2$ represents all positive integers (Lagrange $\sim$ 1770).

Q is **universal** if it represents all the numbers in $\mathbb{N}$.

Example:
(1) $Q(x, y, z, t) = x^2 + y^2 + z^2 + t^2$ is universal (Lagrange),
(2) $Q(x, y, z) = x^2 + y^2 + z^2$ is **not** universal ($x^2 + y^2 + z^2 \neq 7$).
(3) many indefinite quadratic forms, eq. $x^2 - y^2 - dz^2$, $4 \nmid d$.

We assume that Q is positive definite.

Q is **universal** if it represents all the numbers in $\mathbb{N}$ and is positive definite.

$Q = \sum_{1 \le i \le j \le r} a_{ij} x_i x_j$ is **classical** if a_{ij} is even for $i \neq j$.

Theorem (Conway-Schneeberger, 1995). Let Q be classical. Then Q is universal if and only if it represents $\{1, 2, 3, 5, 6, 7, 10, 14, 15\}$.

Proof. Q - universal $\Longrightarrow$ Q - represents 1 $\Longrightarrow$ $Q = x^2 + \cdots$
Q - represents 2 $\Longrightarrow$ $Q = x^2 + 2y^2 + 2axy + \cdots$
Q - positive definite $\Longrightarrow$ $0 \le \det \begin{pmatrix} 1 & a \\ a & 2 \end{pmatrix} = 2 - a^2 \Longrightarrow a \in \{0, 1\}$
Hence, $Q = x^2 + 2y^2 + \cdots$ or
$Q = x^2 + 2xy + 2y^2 + \cdots = (x + y)^2 + y^2 + \cdots = X^2 + Y^2 + \cdots$
and so on. $\square$

Theorem (Bhargava-Hanke, 2011). Q is universal if and only if Q represents $\{1, \dots, 290\}$.

Example: $x^2 + 2y^2 + 5z^2 + 5t^2$ represents $\mathbb{N} \setminus \{15\}$.

Quadratic form: $Q(x_1, \ldots, x_r) = \sum_{1 \leq i \leq j \leq r} a_{ij} x_i x_j, \quad a_{ij} \in \mathbb{Z}$.
r is the rank of Q.
Q is universal if it represents all the numbers in $\mathbb{N}$ and is positive definite.
Q is classical if $a_{ij} \in 2\mathbb{Z}$ for $i \neq j$.

$$D > 1 \text{ squarefree, } \omega_D := \begin{cases} \sqrt{D}, & D \equiv 2, 3 \pmod 4 \\ \frac{1+\sqrt{D}}{2}, & D \equiv 1 \pmod 4. \end{cases}$$

$$\mathbb{Q} \quad \rightsquigarrow \quad K := \mathbb{Q}(\sqrt{D}),$$

$$\mathbb{Z} \quad \rightsquigarrow \quad \mathcal{O}_K := \mathbb{Z}[\omega_D],$$

$$\mathbb{N} \quad \rightsquigarrow \quad \mathcal{O}_K^+ := \left\{ a + b\sqrt{D} \in \mathcal{O}_K \mid a + b\sqrt{D} > 0,\ a - b\sqrt{D} > 0 \right\}.$$

Example: $1 + \sqrt{2} \notin \mathcal{O}^+_{\mathbb{Q}(\sqrt{2})}$, $2 - \sqrt{2} \in \mathcal{O}^+_{\mathbb{Q}(\sqrt{2})}$.

Quadratic form: $Q(x_1, \ldots, x_r) = \sum_{1 \leq i \leq j \leq r} a_{ij} x_i x_j, \quad a_{ij} \in \mathcal{O}_K$.

Q is **totally positive definite** if $Q(\mathbf{x}) \in \mathcal{O}_K^+$ for all $\mathbf{x} \in \mathcal{O}_K^r \setminus \{\mathbf{0}\}$.

Q is **universal** if it:
(1) represents all the numbers in $\mathcal{O}_K^+$,
(2) is totally positive definite.

Q is **classical** if $a_{ij} \in 2\mathcal{O}_K$ for all $i \neq j$.

$D > 0$, square free, $\omega_D := \begin{cases} \sqrt{D}, & D \equiv 2,3 \pmod 4 \\ \frac{1+\sqrt{D}}{2}, & D \equiv 1 \pmod 4). \end{cases}$, $K := \mathbb{Q}(\sqrt{D})$, $\mathcal{O}_K := \mathbb{Z}[\omega_D]$,

$\mathcal{O}_K^+ := \left\{ a + b\sqrt{D} \in \mathcal{O}_K \mid a + b\sqrt{D} > 0,\ a - b\sqrt{D} > 0 \right\}$.

Quadratic form: $Q(x_1, \dots, x_r) = \sum_{1 \le i \le j \le r} a_{ij} x_i x_j$, $a_{ij} \in \mathcal{O}_K^+$, r is the rank of Q.

Q is universal if it is totally positive definite and represents all the numbers in $\mathcal{O}_K^+$.
Q is classical if $a_{ij} \in 2\mathcal{O}_K$ for all $i \neq j$.

Universal quadratic forms exist over every K.

Example: If sum of r squares $x_1^2 + \cdots + x_r^2$ is universal over K then:
(1) $K = \mathbb{Q}$ and $r = 4$, or
(2) $K = \mathbb{Q}(\sqrt{5})$ and $r = 3$. (Maass, Siegel, 1945).

Question: For a given K, what is the minimal possible r?

(1) Characterisation of classical universal ternary quadratic forms (exist only in $\mathbb{Q}(\sqrt{2})$, $\mathbb{Q}(\sqrt{3})$ and $\mathbb{Q}(\sqrt{5})$) (Chan, Kim, Raghavan, 1996).
(2) In every $\mathbb{Q}\left(\sqrt{n^2-1}\right)$, $n^2 - 1$ squarefree, there is an universal form with $r = 8$ (Kim, 1999).
(3) There are only finitely many $\mathbb{Q}\left(\sqrt{D}\right)$ with an universal form with $r = 7$ (Kim, Kim, Park, 2021).

$D > 0$, square free, $\omega_D := \begin{cases} \sqrt{D}, & D \equiv 2,3 \pmod 4 \\ \frac{1+\sqrt{D}}{2}, & D \equiv 1 \pmod 4). \end{cases}$, $K := \mathbb{Q}(\sqrt{D})$, $\mathcal{O}_K := \mathbb{Z}[\omega_D]$,

$\mathcal{O}_K^+ := \left\{ a + b\sqrt{D} \in \mathcal{O}_K \mid a + b\sqrt{D} > 0, \ a - b\sqrt{D} > 0 \right\}$.

Quadratic form: $Q(x_1, \dots, x_r) = \sum_{1 \le i \le j \le r} a_{ij} x_i x_j$, $a_{ij} \in \mathcal{O}_K^+$, r is the rank of Q.

Q is universal if it is totally positive definite and represents all the numbers in $\mathcal{O}_K^+$.
Q is classical if $a_{ij} \in 2\mathcal{O}_K$ for all $i \neq j$.

$$R(K) := \min\{ \ r \mid \text{there exists universal q. f. over } K \text{ of rank } r \ \}$$

$$R_{cl}(K) := \min\{r \mid \text{there is classical univ. q. f. over } K \text{ of rank } r\}$$

Theorem (Blomer-Kala, 2015; Kala, 2016). For every R there exist infinitely many squarefree D such that $R(\mathbb{Q}(\sqrt{D})) \geq R$.

Their method produced only a very sparse set of D such that $R(\mathbb{Q}(\sqrt{D})) \geq R$ (containing $\approx e^{-cR}\sqrt{X}$ discriminants $D \leq X$ for some $c > 1$).

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Theorem (Blomer-Kala, 2015; Kala, 2016). For every R there exist infinitely many squarefree D such that $R(\mathbb{Q}(\sqrt{D})) \geq R$.

Let us focus on the case of classical quadratic forms.

Theorem A (Kala-Yatsyna-Ż., 2023) For every R and X let

$$\mathcal{D}(R, X) := \#\{\ D \leq X \text{ squarefree} \mid R_{cl}(\mathbb{Q}(\sqrt{D})) \leq R\ \}.$$

Then for every $X \geq 2^{12}R^{12}(\log X)^4$ we have

$$\mathcal{D}(R, X) < 300R^{3/2}X^{7/8}(\log X)^{3/2}.$$

Theorem B (Kala-Yatsyna-Ż., 2023) Let $\varepsilon > 0$. Then for almost all (in the sense of natural density) squarefree $D > 0$,

$$R(\mathbb{Q}(\sqrt{D})) \geq D^{\frac{1}{24}-\varepsilon}, \qquad R_{cl}(\mathbb{Q}(\sqrt{D})) \geq D^{\frac{1}{12}-\varepsilon}.$$

Proof of Theorem B. Put $R = X^{\frac{1}{12}-\varepsilon}$ is Theorem A. $\square$

$R(K) := \min\{\ r \mid$ there exists universal quadratic form over K of rank $r\ \}$
$R_{cl}(K) := \min\{\ r \mid$ there exists classical universal quadratic form over K of rank $r\ \}$
$\mathcal{D}(R, X) := \#\{\ D \leq X$ squarefree $\mid R_{cl}(\mathbb{Q}(\sqrt{D})) \leq R\ \}$
Theorem A (Kala-Yatsyna-Ż., 2023). For every $X \geq 2^{12}R^{12}(\log X)^4$ we have

$$\mathcal{D}(R, X) < 300R^{3/2}X^{7/8}(\log X)^{3/2}.$$

Proof of Theorem A. Assume that $D \equiv 2, 3 \pmod 4$, $K := \mathbb{Q}(\sqrt{D})$.
Consider the continued fraction expansion of $\sqrt{D}$:

$$\sqrt{D} = u_0 + \cfrac{1}{u_1 + \cfrac{1}{u_2 + \cfrac{1}{u_3 + \cdots}}} =: [u_0; \overline{u_1, \ldots, u_s}].$$

For example: $\sqrt{5} = 2 + (\sqrt{5} - 2) = 2 + \frac{1}{\sqrt{5}+2} = 2 + \frac{1}{4+(\sqrt{5}-2)} = 2 + \cfrac{1}{4+\cfrac{1}{\sqrt{5}+2}} = \ldots = 2 + \cfrac{1}{4+\cfrac{1}{4+\cfrac{1}{4+\cdots}}} = [2; \overline{4}]$.

Denote $u := \max\{u_{2j+1} \mid j \geq 0\} = u_{2i+1}$.

Let $u_0 + \cfrac{1}{u_1 + \cfrac{1}{\cdots + \cfrac{1}{u_k}}} = \frac{p_k}{q_k}$ for every k and

$B_r := (p_{2i-1} + q_{2i-1}\sqrt{D}) + r(p_{2i} + q_{2i}\sqrt{D})$ for $0 \leq r \leq u$.

Then $B_r \in \mathcal{O}_K^+$ and there is $\delta \in K^+$ such that $\mathrm{Tr}(\delta B_r) = 1$ and $\mathrm{Tr}(\delta \mathcal{O}_K) \subseteq \mathbb{Z}$.

$R(K) := \min\{\ r \mid \text{there exists universal quadratic form over } K \text{ of rank } r\ \}$
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Proof of Theorem A. Assume that $D \equiv 2, 3 \pmod 4$, $K := \mathbb{Q}(\sqrt{D})$.

$\sqrt{D} = u_0 + \cfrac{1}{u_1 + \cfrac{1}{u_2 + \cfrac{1}{u_3 + \cdots}}} =: [u_0; \overline{u_1, \ldots, u_s}]$. Denote $u := \max\{u_{2j+1} \mid j \geq 0\} = u_{2i+1}$.

Let $u_0 + \cfrac{1}{u_1 + \cfrac{1}{\cdots + \cfrac{1}{u_k}}} = \frac{p_k}{q_k}$ and $B_r := (p_i + q_i\sqrt{D}) + r(p_{i+1} + q_{i+1}\sqrt{D})$ for $0 \leq r \leq u$.

Then $B_r \in \mathcal{O}_K^+$ and there is $\delta \in K^+$ such that $\mathrm{Tr}(\delta B_r) = 1$ and $\mathrm{Tr}(\delta\mathcal{O}_K) \subseteq \mathbb{Z}$.

Let $\varphi : \mathcal{O}_K^R \ni (a_1 + b_1\sqrt{D}, \ldots, a_R + b_R\sqrt{D}) \mapsto (a_1, b_1, \ldots, a_R, b_R) \in \mathbb{Z}^{2R}$.

Let Q be a classical universal q. f. over $\mathbb{Q}(\sqrt{D})$ of rank R. Let $Q(w_r) = B_r$.

Then $q(v) := \mathrm{Tr}(\delta Q(\varphi^{-1}(v)))$ is classical and positive definite q. f. over $\mathbb{Q}$ of rank $2R$. Moreover,

$$q(\pm\varphi(w_r)) = \mathrm{Tr}(\delta Q(w_r)) = \mathrm{Tr}(\delta B_r) = 1.$$

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$\mathcal{D}(R, X) := \#\{\ D \leq X \text{ squarefree} \mid R_{cl}(\mathbb{Q}(\sqrt{D})) \leq R\ \}$

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Proof of Theorem A. Assume that $D \equiv 2, 3 \pmod 4$, $K := \mathbb{Q}(\sqrt{D})$.

$\sqrt{D} = u_0 + \cfrac{1}{u_1 + \cfrac{1}{u_2 + \cfrac{1}{u_3 + \cdots}}} =: [u_0; \overline{u_1, \ldots, u_s}]$. Denote $u := \max\{u_{2j+1} \mid j \geq 0\} = u_{2i+1}$.

Let $u_0 + \cfrac{1}{u_1 + \cfrac{1}{\cdots + \cfrac{1}{u_k}}} = \frac{p_k}{q_k}$ and $B_r := (p_i + q_i\sqrt{D}) + r(p_{i+1} + q_{i+1}\sqrt{D})$ for $0 \leq r \leq u$.

Then $B_r \in \mathcal{O}_K^+$ and there is $\delta \in K^+$ such that $\mathrm{Tr}(\delta B_r) = 1$ and $\mathrm{Tr}(\delta\mathcal{O}_K) \subseteq \mathbb{Z}$.
Let $\varphi : \mathcal{O}_K^R \ni (a_1 + b_1\sqrt{D}, \ldots, a_R + b_R\sqrt{D}) \mapsto (a_1, b_1, \ldots, a_R, b_R) \in \mathbb{Z}^{2R}$.
Let Q be a classical universal q. f. over $\mathbb{Q}(\sqrt{D})$ of rank R. Let $Q(w_r) = B_r$.
Then $q(v) := \mathrm{Tr}(\delta Q(\varphi^{-1}(v)))$ is classical and positive definite q. f. on $\mathbb{Z}^{2R}$ of rank $2R$.
Moreover, $q(\pm\varphi(w_r)) = \mathrm{Tr}(\delta Q(w_r)) = \mathrm{Tr}(\delta B_r) = 1$.

We have constructed $2(u+1) > 2u$ arguments v such that $q(v) = 1$. Hence, $2u < 2 \cdot 2R$, that is, $u < 2R$.

If $X \geq B^{12}(\log X)^4$ then

$$\#\{\ D \leq X \mid \sqrt{D} = [u_0; \overline{u_1, \ldots, u_s}],\ \max\{u_{2i+1} \mid i \geq 0\} \leq B\ \} < 100B^{3/2}X^{7/8}(\log X)^4.$$

Apply the above inequality with $B = 2R$. $\square$

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Theorem B (Kala-Yatsyna-Ż., 2023) Let $\varepsilon > 0$. Then for almost all (in the sense of natural density) squarefree $D > 0$,

$$R(\mathbb{Q}(\sqrt{D})) \ge D^{\frac{1}{24}-\varepsilon}, \qquad R_{cl}(\mathbb{Q}(\sqrt{D})) \ge D^{\frac{1}{12}-\varepsilon}.$$

Let $m \in \mathbb{N}$.

Q is $m\mathcal{O}_K$-universal if it is totally positive definite and represents $m\mathcal{O}_K^+$.

If Q is universal but is not classical then $2Q$ is $2\mathcal{O}_K$-universal and classical.

$R(K) := \min\{ r \mid \text{there exists universal quadratic form over } K \text{ of rank } r \}$
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$\mathcal{D}(R, X) := \#\{ D \le X \text{ squarefree} \mid R_{cl}(\mathbb{Q}(\sqrt{D})) \le R \}$

Theorem A (Kala-Yatsyna-Ż., 2023). For every $X \ge 2^{12} R^{12} (\log X)^4$ we have

$$\mathcal{D}(R, X) < 300 R^{3/2} X^{7/8} (\log X)^{3/2}.$$

Theorem B (Kala-Yatsyna-Ż., 2023) Let $\varepsilon > 0$. Then for almost all (in the sense of natural density) squarefree $D > 0$,

$$R(\mathbb{Q}(\sqrt{D})) \ge D^{\frac{1}{24} - \varepsilon}, \qquad R_{cl}(\mathbb{Q}(\sqrt{D})) \ge D^{\frac{1}{12} - \varepsilon}.$$

Q is $m\mathcal{O}_K$-universal if it is totally positive definite and represents $m\mathcal{O}_K^+$.
If Q is universal but is not classical then $2Q$ is $2\mathcal{O}_K$-universal and classical.

$\mathcal{D}(R, m, X) := \#\{ D \le X \text{ squarefree} \mid \text{there exists } m\mathcal{O}_K-$ universal classical q. f. over $K = \mathbb{Q}(\sqrt{D})$ of rank $R \}$.

Theorem C (Kala-Yatsyna-Ż., 2023) For every $X \ge B(R, m)^{12} (\log X)^4$ we have

$$\mathcal{D}(R, m, X) < 100 B(R, m)^{3/2} X^{7/8} (\log X)^{3/2},$$

where $B(R, m) := \frac{1}{2} C(2R, m)$, and

$$C(R, m) := \begin{cases} 2R, & m = 1, \\ \max\{480, 2R(R-1)\}, & m = 2, \\ \sum_{k=0}^{R} \binom{R}{k} \frac{\pi^{\frac{k}{2}}}{\Gamma\left(\frac{k}{2}+1\right)} m^{\frac{k}{2}}, & m \ge 3. \end{cases}$$

Thank you for your attention.

arXiv: 2302.12080 (paper by Kala, Yatsyna, Ż.)
arXiv: 2301.13222 (survey article by Kala)